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THE SPINNING ELECTRON

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Dedicated to Professor Per-Olov Löwdin on the Occasion of his 60th Birthday

Synopsis

The properties of the *quantum mechanical rotor* are reviewed, and the conditions studied under which its dynamics is in accordance with the invariance requirements of the inhomogeneous Lorentz group. It is shown that these requirements select the *spin* $^1/_2$ quantum states as the only permissible ones, and that the motion of the rotor must be governed by the *Dirac equation*. The properties of this equation and its solutions are then reconsidered in the new perspective, with special emphasis on the symmetry of the problem. A simple interpretation is given of the *PCT operation*, and Wigner's *time reversal* operation is shown to be composed of two more elementary operations.

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Introduction

The idea that the electron is a spinning top was introduced by Uhlenbeck and Goudsmit (1925) fifty years ago and has been touched upon at numerous occasions ever since. Dirac (1928) found "a great deal of truth in the spinning electron model, at least as a first approximation", but he did not make any attempts to interpret the "other dynamical variables" required "besides the co-ordinates and momenta of the electron". Instead, he created a purely mathematical model of the electron, in which these variables are represented by 4×4 matrices. The resulting equation leads to complete agreement with experiment and is, therefore, the equation of motion for the electron.

Several authors have felt the need of some type of interpretation of the internal variables in Dirac's theory, and have explored the quantum theory of rotating systems with this in mind. Thus, Bopp and Haag (1950) drew attention to the fact, that the differential operators describing the angular momentum of a two particle system admit eigenfunctions with half-integral quantum numbers. Yet, they found that no associated Schrödinger equation could make use of these through its regular solutions.

These findings, together with the generally accepted view that it is impossible to formulate a satisfactory relativistic description of a 3-dimensional rotor, have led to the consideration of more complex models with added degrees of freedom. At the same time, the scope has been widened by extending the group of particle characteristics to be described to include e.g. isospin and hypercharge. Allcock (1961), in his investigations, considers a particle model based on two 3-dimensional rotors rotating with respect to each other. Other authors (van Winter, 1957; Hillion et Vigiér, 1958; Bohm, Hillion and Vigiér, 1960) consider instead a 4-dimensional space-time rotor. A comprehensive list of the many diverse classical and quantum mechanical papers in the field is presented in the monograph by Corben (1968).

A study of the various sophisticated models does admittedly leave one with the impression, that the whole field has acquired a somewhat metaphysical character. It is at least fair to say that no simple alternative to Dirac's purely mathematical model af the electron has emerged.

The alternative does exist, however, as we show in the present paper. It is in fact nothing but the elementary 3-dimensional rotor governed by relativistic quantum mechanics. The dynamics of the rotor is in all respects identical with the dynamics of a DIRAC particle, and hence it gives us new and equally exact ways of visualizing the sometimes rather complex behaviour of electrons.

To make the following presentation reasonably self-contained we summarize the most relevant properties of a 3-dimensional rotor in section 2. Section 3 discusses the relativistic description of a spinless particle; the extension to the relativistic rotor as a model of a particle with spin is considered in section 4, and the possible forms of a local Hamiltonian are derived in section 5. In agreement with DIRAC's conclusions, it is found that only for $s = \frac{1}{2}$ can one construct a local relativistic Hamiltonian (the DIRAC Hamiltonian), and the rotor is in this case an asymmetric top. The DIRAC equation and its solutions are then discussed in sections 6–12 in the light of the preceding sections. The invariance group of the problem is described, and detailed expressions are given for all symmetry operations of this group. Throughout the paper we operate with an unassigned indicator, reflecting the fact that the basic commutator relations may be written in two ways, either with an i or a-i.

2. The quantum mechanical rotor

Consider a right handed Cartesian coordinate system S_0, with axes X, Y, Z and origin O. Two points

$$\boldsymbol{r}_1 = (x_1, y_1, z_1), \qquad \boldsymbol{r}_2 = (x_2, y_2, z_2) \tag{1}$$

define a second right handed system S with origin O and axes specified by the unit vectors

$$\left.\begin{aligned} \boldsymbol{e}_1 &= \frac{1}{2\sin\frac{u}{2}}\left(\frac{\boldsymbol{r}_1}{r_1} - \frac{\boldsymbol{r}_2}{r_2}\right), \\ \boldsymbol{e}_2 &= \frac{1}{2\cos\frac{u}{2}}\left(\frac{\boldsymbol{r}_1}{r_1} + \frac{\boldsymbol{r}_2}{r_2}\right), \\ \boldsymbol{e}_3 &= \frac{1}{r_1 r_2 \sin u}\,\boldsymbol{r}_1 \times \boldsymbol{r}_2 \equiv \boldsymbol{e}_1 \times \boldsymbol{e}_2, \end{aligned}\right\} \tag{2}$$

where u is the angle between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. The orientation of S with respect to S_0 may be specified by three Euler angles α, β, γ such that (Rose 1957) S is obtained from S_0 by

1) a rotation about the Z-axis through the angle α,
2) a rotation about the new Y-axis through the angle β,
3) a rotation about the new Z-axis through the angle γ.

The following relations are then valid:

$$\left.\begin{aligned} x_1 &= -r_1\left[\cos\alpha\cos\beta\sin\left(\gamma-\frac{u}{2}\right)+\sin\alpha\cos\left(\gamma-\frac{u}{2}\right)\right],\\ y_1 &= -r_1\left[\sin\alpha\cos\beta\sin\left(\gamma-\frac{u}{2}\right)-\cos\alpha\cos\left(\gamma-\frac{u}{2}\right)\right],\\ z_1 &= r_1\sin\beta\sin\left(\gamma-\frac{u}{2}\right),\end{aligned}\right\}\tag{3}$$

$$\left.\begin{aligned} x_2 &= -r_2\left[\cos\alpha\cos\beta\sin\left(\gamma+\frac{u}{2}\right)+\sin\alpha\cos\left(\gamma+\frac{u}{2}\right)\right],\\ y_2 &= -r_2\left[\sin\alpha\cos\beta\sin\left(\gamma+\frac{u}{2}\right)-\cos\alpha\cos\left(\gamma+\frac{u}{2}\right)\right],\\ z_2 &= r_2\sin\beta\sin\left(\gamma+\frac{u}{2}\right).\end{aligned}\right\}\tag{4}$$

The components s_1, s_2, s_3 of the vector operator

$$\boldsymbol{s} = -\iota\hbar\left[\boldsymbol{r}_1\times\nabla_1+\boldsymbol{r}_2\times\nabla_2\right]\tag{5}$$

are the generators for rotations of S about the X, Y, Z axes, respectively. A finite rotation through an angle ε about a unit vector $\boldsymbol{n}$ is effected by the operator

$$Q(\boldsymbol{n},\varepsilon) = \exp(-\iota\varepsilon\boldsymbol{n}\cdot\boldsymbol{s}/\hbar).\tag{6}$$

The "indicator" ι is either i or $-i$, with i being the ordinary imaginary unit. Obviously, $Q(\boldsymbol{n},\varepsilon)$ is independent of the value assigned to the indicator.

Substitution of (3) and (4) into (5) gives:

$$\left.\begin{aligned} s_1 &= \iota\hbar\left(\sin\alpha\frac{\partial}{\partial\beta}+\cot\beta\cos\alpha\frac{\partial}{\partial\alpha}-\frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right),\\ s_2 &= \iota\hbar\left(-\cos\alpha\frac{\partial}{\partial\beta}+\cot\beta\sin\alpha\frac{\partial}{\partial\alpha}-\frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right),\\ s_3 &= -\iota\hbar\frac{\partial}{\partial\alpha}. \end{aligned}\right\} \tag{7}$$

The operators

$$\zeta_1 = \boldsymbol{s}\cdot\boldsymbol{e}_1,\qquad \zeta_2 = \boldsymbol{s}\cdot\boldsymbol{e}_2,\qquad \zeta_3 = \boldsymbol{s}\cdot\boldsymbol{e}_3 \tag{8}$$

commute with every component of $\boldsymbol{s}$ and have the form:

$$\left.\begin{aligned} \zeta_1 &= \iota\hbar\left(-\sin\gamma\frac{\partial}{\partial\beta}-\cot\beta\cos\gamma\frac{\partial}{\partial\gamma}+\frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha}\right),\\ \zeta_2 &= \iota\hbar\left(-\cos\gamma\frac{\partial}{\partial\beta}+\cot\beta\sin\gamma\frac{\partial}{\partial\gamma}-\frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha}\right),\\ \zeta_3 &= -\iota\hbar\frac{\partial}{\partial\gamma}. \end{aligned}\right\} \tag{9}$$

They satisfy the "anomalous" commutator relations

$$[\zeta_i,\zeta_j] = -\iota\hbar\varepsilon_{ijk}\zeta_k, \tag{10}$$

whereas the operators s_1, s_2, s_3 satisfy the "normal" relations

$$[s_i,s_j] = \iota\hbar\varepsilon_{ijk}s_k. \tag{11}$$

ε_{ijk} is the Levi-Civita symbol, antisymmetric in all three indices ($\varepsilon_{123} = 1$), and the convention of summing over repeated indices is understood.

We also note, that

$$s^2 = -\hbar^2\left[\frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\left(\sin\beta\frac{\partial}{\partial\beta}\right)+\frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\gamma^2}\right)-\frac{2\cos\beta}{\sin^2\beta}\frac{\partial^2}{\partial\alpha\partial\gamma}\right], \tag{12}$$

where

$$s^2 = s_is_i = \zeta_i\zeta_i. \tag{13}$$

The expressions (5) – (12) are, of course, well known. They are reproduced here for the sake of reference and in order to stress, that the s_i and ζ_i operate directly on the "dreibein" defined by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, or equivalently, on functions depending on the orientation of the dreibein through α, β and γ. Thus, we do not consider $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ as coordinates of particles, they are merely mathematical points by means of which the dreibein may be defined. r_1, r_2 and u are, in accordance with this, dummy coordinates which drop out of the description as soon as the Euler angles are introduced. It is in this way that it becomes possible to separate the fact that a system may have an orientation, from more or less arbitrary speculations concerning an internal distribution of matter. That such a separation can be made is, of course, the basic assumption behind most efforts mentioned in the Introduction — with the work of Bopp and Haag as an exception.

The vector $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ may play a very different role in other contexts, as in the theory of two-electron atoms (Hylleraas, 1929; Breit, 1930) where they do represent particle coordinates. r_1, r_2 and u are then actual internal variables, of the greatest importance for the character of the atomic states. The construction of internal coordinate systems similar to ours has consequently been studied by several authors. A review is due to Bhatia and Temkin (1964).

Let us now assume that the dreibein discussed above describes the orientation of an elementary particle with respect to S_0. The probability amplitude for this orientation is then a wavefunction built over the simultaneous eigenfunctions $D^s_{mn}(\alpha, \beta, \gamma)$ of the commuting operators s^2, s_3 and ζ_3. These eigenfunctions have been known since the early days of quantum mechanics, and up-to-date presentations of their properties, as well as the various phase conventions introduced in the course of time, may be found in the books by Bohr and Mottelson (1969) and Judd (1975). They satisfy the relations:

$$\left.\begin{aligned} s^2 D^s_{mn} &= s(s+1)\hbar^2 D^s_{mn} && (s = 0, \tfrac{1}{2}, 1 \ldots), \\ s_3 D^s_{mn} &= m\hbar D^s_{mn} && (m = s, s-1, \ldots, -s), \\ \zeta_3 D^s_{mn} &= n\hbar D^s_{mn} && (n = s, s-1, \ldots, -s). \end{aligned}\right\} \quad (14)$$

For each value of s they define a linear function space Ω_s of dimension $(2s+1)^2$. Properly normalized they satisfy the orthonormality condition:

$$\left.\begin{aligned} \langle D^s_{mn} \mid D^{s'}_{m'n'} \rangle &= \int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta d\beta \int_0^{4\pi} d\gamma D^s_{mn}(\alpha, \beta, \gamma)^* D^{s'}_{m'n'}(\alpha, \beta, \gamma) \\ &= \delta_{ss'}\delta_{mm'}\delta_{nn'}, \end{aligned}\right\} \quad (15)$$

and the phases may be chosen such that

$$\left.\begin{aligned} (s_1 \pm \iota s_2) D^s_{mn} &= \hbar\left[(s \mp m)(s \pm m + 1)\right]^{1/2} D^s_{m\pm 1, n}, \\ (\zeta_1 \mp \iota\zeta_2) D^s_{mn} &= \hbar\left[(s \mp n)(s \pm n + 1)\right]^{1/2} D^s_{m, n\pm 1}. \end{aligned}\right\} \quad (16)$$

Thus we have, for $s = \frac{1}{2}$:

$$\left.\begin{aligned} \theta_1 &= D^{1/2}_{1/2,\,1/2} = (8\pi^2)^{-1/2}\cos\frac{\beta}{2}\,e^{\iota\alpha/2}\,e^{\iota\gamma/2}, \\ \theta_2 &= D^{1/2}_{-1/2,\,1/2} = (8\pi^2)^{-1/2}\sin\frac{\beta}{2}\,e^{-\iota\alpha/2}\,e^{\iota\gamma/2}, \\ \theta_3 &= D^{1/2}_{1/2\,-1/2} = -(8\pi^2)^{-1/2}\sin\frac{\beta}{2}\,e^{\iota\alpha/2}\,e^{-\iota\gamma/2}, \\ \theta_4 &= D^{1/2}_{-1/2,\,-1/2} = (8\pi^2)^{-1/2}\cos\frac{\beta}{2}\,e^{-\iota\alpha/2}\,e^{-\iota\gamma/2}. \end{aligned}\right\} \quad (17)$$

It was shown by EULER, in his pioneer work on the motion of rigid bodies two hundred years ago, that the configuration space for a 3-dimensional rotor is the 4-dimensional unit sphere (see, e.g. WHITTAKER, 1904), each orientation of the rotor corresponding to two points on the sphere. The functions D^s_{mn} may accordingly be viewed as 4-dimensional spherical harmonics (HUND, 1928), and Ω_s is an irreducible function space under the operations of $O(4)$, the 4-dimensional orthogonal group. The operators s_i and ζ_i represent the generators of $O(4)$. It is for certain purposes convenient to replace them by the operators

$$\left.\begin{aligned} \lambda_i &= s_i - \zeta_i, \\ \chi_i &= s_i + \zeta_i, \end{aligned}\right\} \quad (18)$$

which obey the commutator relations:

$$\left.\begin{aligned} [\lambda_i, \lambda_j] &= \iota\hbar\varepsilon_{ijk}\lambda_k, \\ [\lambda_i, \chi_j[&= \iota\hbar\varepsilon_{ijk}\chi_k, \\ [\chi_i, \chi_j] &= \iota\hbar\varepsilon_{ijk}\lambda_k. \end{aligned}\right\} \quad (19)$$

Having characterized the functions from which the probability amplitude for the orientation of a 3-dimensional rotor may be constructed, we shall pass on to a discussion of its dynamics. Our basic assumption will be, that it is possible to construct a Hamiltonian of the form

$$H = H(s_1, s_2, s_3; \zeta_1, \zeta_2, \zeta_3; a), \quad (20)$$

with a referring to a set of external variables which commute with the internal variables s_i and ζ_i. It follows, that

$$[H, s^2] = 0, \quad (21)$$

and hence that the eigenfunctions of H may be written as

$$\psi_k^s = \sum_{i=1}^{(2s+1)^2} \psi_{i,k}\theta_i, \tag{22}$$

where θ_i ($i = 1, 2, \ldots, (2s+1)^2$) are the functions $D^s_{mn}(\alpha, \beta, \gamma)$, and $\psi_{i,k}$ are functions of the external variables.

Each function space Ω_s will thus give rise to its own set of eigenfunctions. Inded, it will turn out that the very form of H will depend on the quantum number s, and that only for $s = \frac{1}{2}$ is it possible to construct a local Hamiltonian. These results are consequences of the constraints imposed by the theory of special relativity, and discussed in the following section.

3. Relativistic description of a spinless particle

The special theory of relativity requires that the laws of physics be invariant under the operations of the inhomogeneous LORENTZ group. Let us, by way of introduction, sketch the implications of this requirement in the case of a free particle without spin.

With

$$x_\mu = (x_1, x_2, x_3, \text{ict}) \tag{23}$$

denoting a general space-time point, we introduce the operators

$$p_\mu = -\iota\hbar \frac{\partial}{\partial x_\mu} \tag{24}$$

and

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu. \tag{25}$$

The following commutator relations are then valid:

$$[x_\mu, p_\nu] = \iota\hbar\delta_{\mu\nu}, \tag{26}$$

$$[p_\mu, p_\nu] = 0, \tag{27}$$

$$[L_{\mu\nu}, L_{\varkappa\lambda}] = \iota\hbar(\delta_{\mu\varkappa}L_{\nu\lambda} + \delta_{\nu\lambda}L_{\mu\varkappa} - \delta_{\nu\varkappa}L_{\mu\lambda} - \delta_{\mu\lambda}L_{\nu\varkappa}), \tag{28}$$

$$[L_{\mu\nu}, p_\lambda] = \iota\hbar(\delta_{\mu\lambda}p_\nu - \delta_{\nu\lambda}p_\mu). \tag{29}$$

We adopt the convention that greek indices take on the values from 1 to 4, italic indices the values from 1 to 3.

The operators $L_{\mu\nu}$ represent the generators of $O(4)$. They are antisymmetric in μ and ν, and hence one introduces new operators which are all independent, viz.

$$\left.\begin{aligned} l_i &= \tfrac{1}{2}\varepsilon_{ijm} L_{jm}, \\ k_i &= L_{i4}. \end{aligned}\right\} \quad (30)$$

The relations (28) are then replaced by

$$\left.\begin{aligned} [l_i, l_j] &= \iota\hbar\varepsilon_{ijm} l_m, \\ [l_i, k_j] &= \iota\hbar\varepsilon_{ijm} k_m, \\ [k_i, k_j] &= \iota\hbar\varepsilon_{ijm} l_m, \end{aligned}\right\} \quad (31)$$

which are similar to (19).

Next, we define operators for finite transformations:

$$\left.\begin{aligned} F(\boldsymbol{a}) &= \exp(-\iota a_i p_i/\hbar), \\ U(\tau) &= \exp(-\iota\tau p_4/\hbar), \\ R(\boldsymbol{n}, \varepsilon) &= \exp(-\iota\varepsilon_i l_i/\hbar), \\ \Lambda(\boldsymbol{n}', \eta) &= \exp(-\iota\eta_i k_i/\hbar), \end{aligned}\right\} \quad (32)$$

characterized by the six real parameters a_i, ε_i, and the four imaginary parameters τ and η_i. $\boldsymbol{n}$ and $\boldsymbol{n}'$ are real unit vectors such that $\varepsilon\boldsymbol{n} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $\eta\boldsymbol{n}' = (\eta_1, \eta_2, \eta_3)$. F generates a spatial translation $\boldsymbol{a}$, U a time displacement $\tau/(ic)$, and R a rotation through the angle ε about $\boldsymbol{n}$. Λ generates a LORENTZ transformation in the direction $\boldsymbol{n}'$ corresponding to the relative velocity V such that

$$\tan\eta = iV/c. \quad (33)$$

The transformations are all independent of the value assigned to the indicator ι.

All F and U and products thereof represent the group $\mathscr{T}$ of translations in MINKOWSKI space. All R and Λ and products thereof represent the proper, orthochronous, homogeneous LORENTZ group $\mathscr{L}_0$. The semidirect product of $\mathscr{T}$ and $\mathscr{L}_0$ is the proper, orthochronous, inhomogeneous LORENTZ group $\mathscr{T}\mathscr{L}_0$. The representations of these groups, as well as of the extensions obtained by adding the operators for space and time inversion, have been thoroughly studied. We refer to papers by WIGNER (1939), BARGMANN and WIGNER (1948), and to the books by ROMAN (1960), LYUBARSKII (1960), and LOMONT (1959).

A representation of $\mathscr{T}\mathscr{L}_0$ for a single particle without spin is obtained by constructing a linear function space which is invariant under the operators (24) and (25). As basic functions we may choose eigenfunctions of the commuting operators p_1, p_2, p_3, and p_4, i.e. functions of the general form

$$|\pi_1, \pi_2, \pi_3, i\pi_0\rangle = \exp(\iota\boldsymbol{\pi}\cdot\boldsymbol{r}/\hbar)\exp(-\iota c\pi_0 t/\hbar). \tag{34}$$

The operator

$$p_\mu p_\mu = \hbar^2\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \tag{35}$$

commutes with all operators in (24) and (25) and hence with all operators in the set (32). A function space which is irreducible under the operators representing $\mathscr{T}\mathscr{L}_0$ will consequently be characterized by a single eigenvalue of $p_\mu p_\mu$. Since

$$p_\mu p_\mu |\boldsymbol{\pi}, i\pi_0\rangle = (\pi^2 - \pi_0^2)|\boldsymbol{\pi}, i\pi_0\rangle, \tag{36}$$

we have the requirement:

$$\pi_0^2 = \pi^2 + m_0^2 c^2, \tag{37}$$

with m_0 being a constant. This constant is identified with the mass of the particle. We further identify $\boldsymbol{\pi}$ with the momentum and $c\pi_0$ with the kinetic energy:

$$E_{kin} = c\pi_0. \tag{38}$$

m_0 and π_0 are, accordingly, assumed to be non-negative; $(\boldsymbol{\pi}, i\pi_0)$ is a time-like four-vector, and

$$\pi_0 = \sqrt{\pi^2 + m_0^2 c^2}. \tag{39}$$

It is easy to verify that all functions of the form (34), with the same m_0, may be generated from one function in the set by use of the operators R and Λ of (32). A convenient choice for the representative function is

$$|0, 0, 0, im_0 c\rangle = \exp(-\iota m_0 c^2 t/\hbar) \qquad \text{when} \quad m_0 > 0, \tag{40}$$

$$|0, 0, 1, i > \exp[\iota(x_3 - ct)/\hbar] \qquad \text{when} \quad m_0 = 0. \tag{41}$$

We get, for instance:

$$\left.\begin{aligned} \Lambda(0,0,1,\eta)|0,0,0,im_0 c\rangle &= \exp(\iota\pi x_3/\hbar)\exp(-\iota c\sqrt{\pi^2 + m_0^2 c^2}\, t/\hbar) \\ &= |0, 0, \pi, i\pi_0\rangle, \end{aligned}\right\} \tag{42}$$

with

$$\tan\eta = i\pi/\sqrt{\pi^2 + m_0^2 c^2} = ic\pi/E_{kin}. \tag{43}$$

By comparing with (33) we obtain the usual expression for the velocity of the particle:

$$V = c^2\pi/E_{kin}. \tag{44}$$

Similarly we get:

$$\Lambda(0,0,1,\eta)|0,0,1,i\rangle = \exp[\iota\pi'(x_3 - ct)/\hbar] = |0,0,\pi',i\pi'\rangle, \tag{45}$$

where

$$\pi' = \exp(-i\eta). \tag{46}$$

Let us now consider the equation of motion for a free spinless particle. The existence of such an equation is, of course, a necessary condition for being able to predict the future from the instantaneous situation. An equation of motion must have the form

$$H\psi = \iota h\frac{\partial\psi}{\partial t}, \tag{47}$$

with ψ being the wavefunction and H a time independent operator, the Hamiltonian of the particle. H and $\iota h\frac{\partial}{\partial t}$ are thus required to be equivalent operators, and this implies that the relations (27)–(29) must remain unaffected by the substitution

$$p_4 \to \frac{i}{c}H. \tag{48}$$

The relations (27)–(29), with the substitution (48), represent what has been called by Dirac "relativistic dynamics in the instant form" (Dirac, 1949). The problem of constructing a dynamical theory is tantamount to finding an H that will satisfy the substituted relations.

The operators $\pm c\sqrt{p^2 + m_0^2c^2}$, with m_0 being an arbitrary constant, will satisfy the relations in our case. m_0 is again fixed as the mass of the particle, and a comparison with (34) and (47) shows, that we must choose

$$H = c\sqrt{p^2 + m_0^2c^2}. \tag{49}$$

The solution for the Hamiltonian is thus unique. Its eigenvalues represent the kinetic energy according to (38).

4. The relativistic rotor

We shall now extend the treatment of the previous section to the case of a quantum mechanical rotor, as a model of a particle for which it is possible to talk about an orientation in space. The coordinates x_μ define the position of the particle by specifying the origin of the coordinate system S_0, in space and time. The EULER angles α,β,γ specify the orientation of the particle, i.e. the orientation of S with respect to S_0.

A first necessary condition for being able to construct a relativistic dynamics for the rotor is the existence of an algebra similar to the one given through the relations (27)–(29). The four-momentum (24) is defined as before, but the operators $L_{\mu\nu}$ must be supplemented with operators built over the internal generators s_i and ζ_i, as given by (7) and (9). Thus we define:

$$J_{\mu\nu} = L_{\mu\nu} + s_{\mu\nu}, \tag{50}$$

and similar to (30):

$$\left.\begin{aligned} J_i &= \tfrac{1}{2}\varepsilon_{ijm} J_{jm}, & s_i &= \tfrac{1}{2}\varepsilon_{ijm} s_{jm}, \\ K_i &= J_{i4}, & \varkappa_i &= s_{i4}. \end{aligned}\right\} \tag{51}$$

The operators $J_{\mu\nu}$ and p_μ must satisfy the relations (27)–(29) with $J_{\mu\nu}$ substituted for $L_{\mu\nu}$. The operators (51) must satisfy relations similar to (31), in particular:

$$\left.\begin{aligned} [s_i, s_j] &= \iota\hbar\varepsilon_{ijm} s_m, \\ [s_i, \varkappa_j] &= \iota\hbar\varepsilon_{ijm} \varkappa_m, \\ [\varkappa_i, \varkappa_j] &= \iota\hbar\varepsilon_{ijm} s_m. \end{aligned}\right\} \tag{52}$$

We have already, by (18), constructed a set of operators satisfying (52), but they cannot be used for the present purpose, because it is essential that the s_i in (52) be identical with the s_i in (7). This is dictated by the form of the rotation operator (6).

With the s_i fixed by this requirement, it only remains to determine the $\varkappa_i$. The second of the relations (52) shows that $\boldsymbol{\kappa}$ must be equal to $\boldsymbol{s}$ times an operator b commuting with $\boldsymbol{s}$:

$$\boldsymbol{\kappa} = b\boldsymbol{s}, \tag{53}$$

and because of (13) we may take this b to be a function of the ζ_i alone. The third of the relations (52) shows finally that the condition

$$b^2 = 1 \tag{54}$$

must hold for b.

In looking for an operator that will satisfy (54) one must exclude the trivial solutions $b = \pm 1$, since $\boldsymbol{s}$ and $\boldsymbol{\kappa}$ must be linearly independent. This implies, that it is impossible to find a universal expression for b, but with (20) and (22) in mind it becomes meaningful to solve (54) within each function space Ω_s separately (cf. (14)). In this way one obtains the following semigeneral solution, independent of the value assigned to ι:

$$\left.\begin{aligned} b &= \exp(-\pi\iota \boldsymbol{e}\cdot\boldsymbol{s}/\hbar) && \text{for } s \text{ integer,} \\ b &= i\exp(-\pi\iota \boldsymbol{e}\cdot\boldsymbol{s}/\hbar) && \text{for s half-integer,} \end{aligned}\right\} \tag{55}$$

with $\boldsymbol{e}$ being an arbitrary unit vector in the internal coordinate system S.

Since no dynamical preference has been given to any of the axes of S so far, we can now introduce such a preference by fixing the direction of $\boldsymbol{e}$. A convenient choice at the present stage is

$$\boldsymbol{e} = \boldsymbol{e}_3. \tag{56}$$

Thus we obtain:

$$\left.\begin{aligned} b &= \frac{2}{\hbar}\frac{\iota}{i}\zeta_3 && \text{for } s = \tfrac{1}{2}, \\ b &= 1 - \frac{2}{\hbar^2}\zeta_3^2 && \text{for } s = 1, \\ &\text{etc.} \end{aligned}\right\} \tag{57}$$

Having determined the operators of the basic algebra we obtain the operators for the finite transformations of $\mathscr{T}\mathscr{L}_0$ by multiplying $R(\boldsymbol{n},\varepsilon)$ in (32) with the operator (6), i.e.

$$Q(\boldsymbol{n},\varepsilon) = \exp(-\iota\varepsilon_i s_i/\hbar). \tag{58}$$

The operator $\Lambda(\boldsymbol{n}',\eta)$ is similarly to be multiplied with

$$\lambda(\boldsymbol{n}',\eta) = \exp(-\iota\eta_i\varkappa_i/\hbar). \tag{59}$$

These operators are again independent of the value assigned to ι.

The operator $p_\mu p_\mu$ of eqn. (35) will also commute with all operators in the new algebra. The irreducible representations of $\mathscr{T}\mathscr{L}_0$ are consequently spanned by functions of the form

$$\psi_j^s = \varphi_j^s(\alpha,\beta,\gamma;\pi,m_0)\,|\,\pi, i\pi_0\rangle \tag{60}$$

where $|\,\pi, i\pi_0\rangle$ is given by (34), and φ_j^s are functions of the internal coordinates, depending parametrically on s, π and m_0. The relation (39) is still valid and the energy is given by (38) as before.

Let us assume, in what follows, that $m_0 \neq 0$. The form of φ_j^s is then completely given, once it is known for $\pi = \mathbf{0}$. The relation is:

$$\varphi_j^s(\alpha,\beta,\gamma;\pi,m_0) = \lambda(\pi/\pi,\eta)\,\varphi_j^s(\alpha,\beta,\gamma;\mathbf{0},m_0), \tag{61}$$

with η given by (43). We are thus left with the problem of classifying $\varphi_j^s(\alpha,\beta,\gamma;\mathbf{0},m_0)$ further with respect to the symmetry of $\mathscr{T}\mathscr{L}_0$.

At this point we note that there is another operator besides $p_\mu p_\mu$ which commutes with all operators in the basic algebra, namely $w_\mu w_\mu$, where

$$w_\mu = (\boldsymbol{p} \times \kappa + p_4 \boldsymbol{s}, -\boldsymbol{p} \cdot \boldsymbol{s}), \tag{62a}$$

(BARGMANN and WIGNER, 1946). USING (53) and (54) we get, that

$$w_\mu w_\mu = p_\mu p_\mu s_i s_i. \tag{62b}$$

This new invariant gives the mathematical justification for the label s in (60).

The components of w_μ do not commute with each other. We have, however, the very important result:

$$[w_\mu, p_\nu] = 0, \tag{63}$$

according to which each ψ_j^s may be taken as an eigenfunction for one of the new w_μ as well. We note, in particular, that ψ_j^s for $\pi = \mathbf{0}$ may be chosen as an eigenfunction of $p_4 s_3$.

For the sake of completeness we also note, that $p_\mu w_\mu$ is an invariant, but since it is identically zero, it is of no use in the present context.

The functions φ_j^s are linear combinations of the $(2s+1)^2$ functions $D^s_{mn}(\alpha, \beta, \gamma)$ of section 2, but it is readily seen that the $2s+1$ functions corresponding to a given value of n constitute an invariant function space under all s_i and $\varkappa_i$. Each value of n will thus give rise to an irreducible representation of $\mathscr{T}\mathscr{L}_0$, with the functions φ_j^s equal to the functions $D^s_{mn}(\alpha, \beta, \gamma)$, $m = s, s-1, \ldots, -s$.

The $2s+1$ irreducible representations ($n = s, s-1, \ldots, -s$) obtained in this way are, however, all equivalent. This follows from general discussions on the irreducible representations of $\mathscr{T}\mathscr{L}_0$ (see references following eqn. (33)), according to which the representative functions for $\boldsymbol{p} = \mathbf{0}$ are characterized as spanning irreducible representations of $R(3)$, the 3-dimensional real rotation group. $R(3)$ is in this context the little group associated with the four-vector $(0, 0, 0, im_0 c)$.

We have thus arrived at the conclusion, that only for $s = 0$ (which is the case already studied in the previous section) is there no redundant degeneracy in the classification of the rotor states When $s \neq 0$ we are left with a $2s+1$ fold degeneracy.

Any function of the form (60) will satisfy the SCHRÖDINGER equation (47) with the Hamiltonian (49). This is, however, of little interest in the present context, since such a Hamiltonian does not effect the internal coordinates at all. We shall consequently look for a more general Hamiltonian by reconsidering the basic algebra and require that it be satisfied with the substitution (48).

The variable $x_4 = ict$ commutes with all operators of the substituted algebra, and may therefore, without loss of generality, be set equal to zero. Thus, we get the substituted operators:

$$\left.\begin{aligned} p_\mu &= \left(p_1, p_2, p_3, \frac{i}{c}H\right), \\ J_i &= l_i + s_i, \\ K_i &= \frac{i}{c}x_i H + \varkappa_i, \end{aligned}\right\} \quad (64)$$

and the algebraic equations involving H become:

$$[H, p_i] = 0, \quad (65)$$

$$[H, J_i] = 0, \quad (66)$$

$$[H, K_i] = \frac{\iota}{i}\hbar c p_i, \quad (67)$$

$$[p_i, K_j] = \frac{\iota}{i}\hbar \delta_{ij}\frac{1}{c}H, \quad (68)$$

$$[J_i, K_j] = \iota\hbar\varepsilon_{ijm}K_m, \quad (69)$$

$$[K_i, K_j] = \iota\hbar\varepsilon_{ijm}J_m. \quad (70)$$

In addition, we have the invariance relation

$$H^2 = c^2 p^2 + m_0^2 c^4. \quad (71)$$

5. The local Hamiltonians

In searching for solutions to the above relations we begin by noting, that (65) and (66) imply that x_1,x_2,x_3 and α,β,γ are cyclic coordinates, i.e. H must be of the form

$$H = H(s_i, \zeta_i, p_i; m_0), \quad (72)$$

as alredy anticipated by (20). The relations (68) and (69) are then automatically satisfied, whereas (67) imposes the conditions

$$\frac{i}{c}[H,\varkappa_j] - \frac{1}{c^2}[H,x_j]H = \iota\hbar p_j. \tag{73}$$

The relation (70) is automatically satisfied whenever (73) is.

The necessary conditions on H are thus contained in (66), (71) and (73).

It follows from (71), as well as from (73), that if H is a polynomial in p_i, then this polynomial must be of the first degree. Any local Hamiltonian must thus be linear in the momentum operators. The only other conceivable solution is the non-local form

$$H = a\sqrt{c^2p^2 + m_0^2c^4},\ a^2 = 1 \tag{74}$$

with a being a function of the s_i and ζ_i.

A short consideration of (66) and (73) shows, that a must commute with every s_i and $\varkappa_i$, and hence the only possible values are ± 1 and $\pm b$, with b given by (55) and (57).

We shall not, however, consider the non-local Hamiltonians further, but instead confine the attention to local Hamiltonians, as the more satisfactory type of operators from a physical point of view.

A local Hamiltonian is, as mentioned above, necessarily linear in the momentum operators. Hence, we write it as

$$H = \lambda + \mu_j p_j, \tag{75}$$

with λ and μ_j being functions of s_i and ζ_i. Insertion in (68) shows, that λ must be a function of the ζ_i alone, and that $\mu_j = \mu s_j$ with μ depending only on the ζ_i. Thus we have:

$$H = \lambda(\zeta_i) + \mu(\zeta_i)(\boldsymbol{s}\cdot\boldsymbol{p}). \tag{76}$$

To determine the functions λ and μ we insert (76) in (73) and compare the coefficients of p_1, p_2 and p_3 in turn. It is then found that a necessary condition for (73) to be satisfied is, that $s_1^2 = s_2^2 = s_3^2 =$ a non-vanishing constant. This is only possible if the operators act in the function space $\Omega_{1/2}$, in which case:

$$\left.\begin{aligned} s_i s_j + s_j s_i &= \tfrac{1}{2}\hbar^2\delta_{ij}, \\ \zeta_i\zeta_j + \zeta_i\zeta_j &= \tfrac{1}{2}\hbar^2\delta_{ij}, \end{aligned}\right\} \tag{77}$$

and

$$\left.\begin{aligned} s_i s_j &= \tfrac{1}{2}\hbar\varepsilon_{ijk}s_k, \\ \zeta_i\zeta_j &= -\tfrac{1}{2}\hbar\varepsilon_{ijk}\zeta_k. \end{aligned}\right\} \tag{78}$$

Hence it follows, that it is impossible to turn a 3-dimensional rotor into a relativistic system with a local Hamiltonian, unless it is endowed with an s quantum number of $\frac{1}{2}$.

We proceed, then, by assuming the validity of (77) and (78). λ and μ are then linear function of $\zeta_1, \zeta_2, \zeta_3$. We shall furthermore deviate from (56) and (57) by choosing $\boldsymbol{e}_1$ as the preferred axis when deflning κ, i.e. we put

$$\boldsymbol{e} = \boldsymbol{e}_1, \tag{79}$$

and hence

$$\kappa = \frac{2}{\hbar} \frac{\iota}{i} \zeta_1 \boldsymbol{s}. \tag{80}$$

Insertion of (77), (78) and (80) in (73) and further comparison of the coefficients of p_1, p_2 and p_3 leads to the unique result

$$\mu = \frac{4c}{\hbar^2} \zeta_1. \tag{81}$$

Finally one obtains, from the terms independent of p_i:

$$\lambda \zeta_1 + \zeta_1 \lambda = 0. \tag{82}$$

This relation shows, in the light of (77), that λ must be a linear combination of ζ_2 and ζ_3, and since no preference has been given to any of the axes perpendicular to $\boldsymbol{e}_1$ we may set

$$\lambda = A \zeta_3, \tag{83}$$

with A being a constant.

Thus we obtain the Hamiltonian

$$H = A \zeta_3 + \frac{4c}{\hbar^2} \zeta_1 (\boldsymbol{s} \cdot \boldsymbol{p}). \tag{84}$$

To determine A we square H and compare with (71), while using (77). This leads to the values

$$A = \pm \frac{2}{\hbar} m_0 c^2, \tag{85}$$

and hence:

$$H = \pm \frac{2}{\hbar} m_0 c^2 \zeta_3 + \frac{4c}{\hbar^2} \zeta_1 (\boldsymbol{s} \cdot \boldsymbol{p}). \tag{86}$$

The eigenfunctions of H are of the form (22) with $s = \frac{1}{2}$, i.e.

$$\psi_k^{1/2}(\boldsymbol{r}, \alpha, \beta, \gamma) = \sum_{i=1}^{4} \psi_{i,k}(\boldsymbol{r})\theta_i(\alpha, \beta, \gamma), \tag{87}$$

where θ_1, θ_2, θ_3, θ_4, are the functions specified by (17). The equation of motion is of the form (47). It becomes identical with the DIRAC equation when transformed to a matrix representation.

6. The Dirac equation

The transformation mentioned is obtained by substituting the general expansion

$$\Psi = \sum_{i=1}^{4} \psi_i(\boldsymbol{r}, t)\theta_i(\alpha, \beta, \gamma) \tag{88}$$

into the equation of motion (47), i.e.

$$H\Psi = \iota\hbar \frac{\partial \Psi}{\partial t}, \tag{47}$$

with H as given by (86). The inner product is then formed with θ_1, θ_2, θ_3, θ_4 in turn and the orthonormality relations (15) utilized. As a result one obtains:

$$(\pm m_0 c^2 \beta + c\alpha \cdot \boldsymbol{p})\psi = \iota\hbar \frac{\partial \psi}{\partial t}, \tag{89}$$

where ψ is a column vector with ψ_1, ψ_2, ψ_3, ψ_4, of (88) as components, and

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \alpha_k = \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix} \quad (k = 1, 2, 3). \tag{90}$$

I is the two-dimensional unit matrix, and

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\iota \\ \iota & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{91}$$

become the PAULI spin matrices when ι is assigned the value i.

Eqn. (89), with the upper sign in front of $m_0 c^2 \beta$ and $\iota = i$, is in fact the DIRAC equation in its Hamiltonian form. The ambiguity in sign of the first term will be commented on in section 13. Until then we shall adopt the plus sign in (86), and write

$$H = m_0 c^2 \zeta_3' + c\zeta_1'(\boldsymbol{s}' \cdot \boldsymbol{p}), \tag{92}$$

where the primed operators, introduced for simplicity, are equal to the corresponding unprimed ones, multiplied by $\frac{2}{\hbar}$.

The present derivation of the DIRAC equation is, of course, rather different from DIRAC's own, since it is based on a model (albeit a very well-defined one) rather than on the purely mathematical properties of hypercomplex numbers. The principles underlying the two derivations are, however, the same, and they may therefore supplement each other in a fruitful way. It is interesting to note, that the 4-dimensional matrices σ_i and ϱ_i occurring in DIRAC's paper (DIRAC, 1928) are nothing but the matrix representatives of our s_i and ζ_i operators multiplied by $\frac{2}{\hbar}$. The sign of ϱ_2 is the opposite of ours, though, and the minus sign in the second of the relations (78) is thus absent in DIRAC's equivalent relation.

We shall now consider the solutions of (47) in the light of the previous sections, with the aim of showing the coherence of our approach. We close the present section with the obvious remark, that the functions (88) are independent of the basis chosen in $\Omega_{1/2}$. In other words: if one prefers to take four orthogonal conbinations of the functions (17) as a new basis, then this has no effect upon the analytical form of Ψ. The matrix representation of (47) will, however, now be different from (89). The fundamental relations (77) and (78) will, on the other hand, be satisfied by the matrices in any representation. This expresses the so-called representation independence of the DIRAC equation.

7. The solutions of the Dirac equation

The solutions of the equation

$$H\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{47}$$

with H given by (92) are, of course, equivalent to the solutions obtained by the more conventional theory, as presented in wellknown textbooks (e.g. BJORKEN and DRELL, 1964; SAKURAI, 1967). Referring to the discussion in section 4 we may present the results of solving (47) in the following way.

The solutions of (47) span two irreducible representations, Γ and $\tilde{\Gamma}$, of $\mathcal{T}\mathcal{L}_0$. These representations become the complex conjugate of each other, when the basis functions spanning them are generated by means of the operators (32), (58) and (59), starting from the two complex conjugate pairs of functions:

$$\left.\begin{aligned}\Phi_1 &= \theta_1(\alpha,\beta,\gamma)\exp(-\iota m_0 c^2 t/\hbar),\\ \Phi_2 &= \theta_2(\alpha,\beta,\gamma)\exp(-\iota m_0 c^2 t/\hbar),\end{aligned}\right\} \tag{93}$$

and

$$\left.\begin{aligned}\tilde{\Phi}_1 &= \theta_4(\alpha,\beta,\gamma)\exp(\iota m_0 c^2 t/\hbar),\\ \tilde{\Phi}_2 &= -\theta_3(\alpha,\beta,\gamma)\exp(\iota m_0 c^2 t/\hbar).\end{aligned}\right\} \tag{94}$$

Let us construct the functions obtained by performing a homogeneous LORENTZ transformation corresponding to the direction

$$\boldsymbol{e} = \boldsymbol{\pi}/\pi \tag{95}$$

and the parameter η given by (43).

The functions exp $(\pm\iota m_0 c^2 t/\hbar)$ are transormed similar to (42). The θ_j functions are transformed by means of the operator

$$\lambda(\boldsymbol{e},\eta) = \exp(-\iota\eta\boldsymbol{e}\cdot\boldsymbol{\kappa}/\hbar), \tag{96}$$

with $\boldsymbol{\kappa}$ given by (80). Introducing the primed operator

$$\boldsymbol{\kappa}' = \frac{2}{\hbar}\boldsymbol{\kappa} = \frac{\iota}{i}\zeta_1'\boldsymbol{s}' \tag{97}$$

this becomes

$$\lambda(\boldsymbol{e},\eta) = \cos\frac{\eta}{2} - i\zeta_1'(\boldsymbol{e}\cdot\boldsymbol{s}')\sin\frac{\eta}{2}. \tag{98}$$

Applying standard trigonometric formulae in connection with the expression (43) for tan η we obtain

$$\left.\begin{aligned}\cos\frac{\eta}{2} &= \left(\frac{|E| + m_0 c^2}{2 m_0 c^2}\right)^{\frac{1}{2}},\\ \sin\frac{\eta}{2} &= i\left(\frac{|E| - m_0 c^2}{2 m_0 c^2}\right)^{\frac{1}{2}},\end{aligned}\right\} \tag{99}$$

where

$$|E| = \sqrt{c^2\pi^2 + m_0^2 c^4}. \tag{100}$$

Thus, we get:

$$\lambda(\boldsymbol{e},\eta) = \cos\frac{\eta}{2}\left[1 + \frac{c}{|E| + m_0 c^2}\zeta_1'(\boldsymbol{\pi}\cdot\boldsymbol{s}')\right]. \tag{101}$$

The functions obtained from (93) and (94) are now readily seen to be

$$\left.\begin{aligned}\Psi_1 &= \quad\cos\frac{\eta}{2}\left[\theta_1+\frac{c}{|E|+m_0c^2}(\pi_z\theta_3+\pi_+\theta_4)\right]\exp(\iota\boldsymbol{\pi}\cdot\boldsymbol{r}/\hbar)\exp(-\iota|E|t/\hbar),\\ \Psi_2 &= \quad\cos\frac{\eta}{2}\left[\theta_2+\frac{c}{|E|+m_0c^2}(\pi_-\theta_3-\pi_z\theta_4)\right]\exp(\iota\boldsymbol{\pi}\cdot\boldsymbol{r}/\hbar)\exp(-\iota|E|t/\hbar),\end{aligned}\right\}\tag{102}$$

and

$$\left.\begin{aligned}\tilde{\Psi}_1 &= \quad\cos\frac{\eta}{2}\left[\theta_4+\frac{c}{|E|+m_0c^2}(\pi_-\theta_1-\pi_z\theta_2)\right]\exp(-\iota\boldsymbol{\pi}\cdot\boldsymbol{r}/\hbar)\exp(\iota|E|t/\hbar),\\ \tilde{\Psi}_2 &= -\cos\frac{\eta}{2}\left[\theta_3+\frac{c}{|E|+m_0c^2}(\pi_z\theta_1+\pi_+\theta_2)\right]\exp(-\iota\boldsymbol{\pi}\cdot\boldsymbol{r}/\hbar)\exp(\iota|E|t/\hbar),\end{aligned}\right\}\tag{103}$$

where

$$\pi_\pm = \pi_1 \pm \iota\pi_2. \tag{104}$$

The functions (102) are eigenfunctions of H and $\boldsymbol{p}$ with eigenvalues $|E|$ and $\boldsymbol{\pi}$, respectively. The functions (103) are eigenfunctions of the same operators with eigenvalues $-|E|$ and $-\boldsymbol{\pi}$.

The function space available for a Dirac particle is the direct sum $\Omega\oplus\tilde{\Omega}$ of the two spaces Ω and $\tilde{\Omega}$, obtained by operating with all operators of the form (32), (58) and (59) on the functions (93) and (94), respectively. A function in Ω represents a particle state, a function in $\tilde{\Omega}$ an antiparticle state. A function with components in both Ω and $\tilde{\Omega}$ represents a superposition of a particle and an antiparticle state.

8. Charge conjugation symmetry

There is a one-to-one correspondence between the functions in the spaces Ω and $\tilde{\Omega}$ specified in the previous section, two corresponding functions being the complex conjugates of each other. This reflects, that whenever a function Ψ is a solution of (47), then the same is true for the complex conjugate function $\tilde{\Psi}$. The process of complex conjugation is thus an invariance operation of the theory. In the following we shall identify this operation, which we denote by C, with the charge conjugation operation of the conventional theory.

The operator effecting the operation C is defined by

$$C_{op}\Psi = \tilde{\Psi} \tag{105}$$

with $\sim$ denoting complex conjugation. It has the obvious property

$$C_{op}^2 = 1. \tag{106}$$

From the explicit expressions (17) we obtain the following relations, already used in passing from (93) to (94):

$$\left.\begin{aligned} \tilde{\theta}_1 &= \theta_4, \\ \tilde{\theta}_2 &= -\theta_3, \\ \tilde{\theta}_3 &= -\theta_2, \\ \tilde{\theta}_4 &= \theta_1. \end{aligned}\right\} \tag{107}$$

Hence, we get for an arbitrary function of the form (88), i.e.

$$\Psi = \theta_1\psi_1 + \theta_2\psi_2 + \theta_3\psi_3 + \theta_4\psi_4, \tag{108}$$

that

$$C_{op}\Psi = \theta_4\psi_1^* - \theta_3\psi_2^* - \theta_2\psi_3^* + \theta_1\psi_4^*, \tag{109}$$

where – in order to facilitate comparisons with the conventional therory – we have used * to denote complex conjugation of a function independent of α, β and γ.

This result may conveniently be written as

$$C\Psi = [\theta_1\theta_2\theta_3\theta_4]\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \psi_1^* \\ \psi_2^* \\ \psi_3^* \\ \psi_4^* \end{bmatrix}. \tag{110}$$

The 4×4 matrix occurring in this relation is readily identified with DIRAC's $-\gamma_2$. It is equal to $i\gamma^2$ in the tensor notation by e.g. BJORKEN and DRELL (1964). Our simple definition (105) of the charge conjugation operation is then seen to coincide with BJORKEN and DRELL's. It differs from e.g. SAKURAI's (1967) in sign. (Definitions in the literature may vary with an arbitrary phase factor).

The operators (32), (58) and (59), from which the operators of $\mathcal{T}\mathcal{L}_0$ are constructed, are all real (see also (101)). This implies that C commutes with all elements of $\mathcal{T}\mathcal{L}_0$. Hence, we may construct the direct product group $\mathcal{C}\times\mathcal{T}\mathcal{L}_0$, where

$$\mathcal{C} = \{E, C\}, \tag{111}$$

E being the identity operation. The function space $\Omega \oplus \tilde{\Omega}$, which contains the totality of solutions of eqn. (47), defines then a single irreducible representation of $\mathscr{C} \times \mathscr{T}\mathscr{L}_0$. This group is thus an invariance group of the theory.

In the following sections we shall augment this invariance group further by adding the space and time inversion operations.

9. Space inversion

The process of space inversion, P, replaces $\boldsymbol{r}$ by $-\boldsymbol{r}$ and thus also $\boldsymbol{p}$ by $-\boldsymbol{p}$. To determine its effect on the internal axes of the rotor it is necessary to go beyond the assumption made in section 2, that the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ of eqn. (1) merely represent mathematical points. We must now assume, that they in some way or other have a physical significance, such that they are replaced by $-\boldsymbol{r}_1$ and $-\boldsymbol{r}_2$ under inversion.

With this assumption it follows from (2), that the directions of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are reversed under P, whereas $\boldsymbol{e}_3$ is left unchanged. The effect on the Euler angles is accordingly:

$$\alpha \to \alpha,\ \beta \to \beta,\ \gamma \to \gamma + \pi. \tag{112}$$

The functions θ_1 and θ_2 in (17) are thus multiplied by ι under inversion, θ_3 and θ_4 are multiplied by $-\iota$. This result is in accordance with the assumption of the conventional theory, that space inversion is effected by the matrix $a\beta$, where β is defined by (90) and a takes one of the four values ± 1, $\pm i$ (see e.g. Bjorken and Drell, 1964; Sakurai, 1967).

Adopting (112) we see from (7), that s_1, s_2 and s_3 are unchanged under inversion. (8), as well as (9), shows that ζ_1 and ζ_2 change sign, whereas ζ_3 remains unaffected.

The relativistic description of a spinless particle is invariant under space inversion, i.e. its symmetry group may be extended from $\mathscr{T}\mathscr{L}_0$ to $\mathscr{T}\mathscr{L}_p$, the orthochronous, inhomogeneous Lorentz group. Within the algebra defined by the operators (24) and (30) P has the following effect:

$$\boldsymbol{p} \to -\boldsymbol{p},\ p_4 \to p_4,\ \boldsymbol{l} \to \boldsymbol{l},\ \boldsymbol{k} \to -\boldsymbol{k}. \tag{113}$$

The substitution (48) requires that

$$H \to H, \tag{114}$$

a condition which is certainly satisfied by the Hamiltonian (49).

The relations (113) and (114) must likewise hold for the generators associated with the relativistic rotor, if we require that P be a symmetry operation in this case as well. In particular, we must require that

$$\boldsymbol{s} \rightarrow \boldsymbol{s}, \ \kappa \rightarrow -\kappa. \tag{115}$$

$\boldsymbol{s}$ is, in fact, unaffected by P. But in order that κ change sign we must require that b, as defined by (53), change sign. ζ_3 is unaffected by P, and the choice (56) is thus an unacceptable one. For $s = \frac{1}{2}$ we must choose b as a linear combination of ζ_1 and ζ_2 alone, as was in fact done in section 5, by (80).

The fact that (56) is an invalid choice, if P is present as a symmetry operation, does not in any sense make the general conclusions of section 4 invalid, since these only refer to the properties of $\mathscr{T}\mathscr{L}_0$ and its representations.

Considering now the requirement (114), we get a narrowing of the condition on λ in passing from (82) to (83), namely that λ must be a constant times ζ_3, in accordance with the actual choice (83).

The Hamiltonian (86) is then unaffected by P, and the description which we have constructed on the basis of section 5 is invariant under space inversion. This remains true also after the inclusion of the charge conjugation operation, since it is evident that C and P commute. We may thus extend the invariance group of the theory from $\mathscr{C} \times \mathscr{T}\mathscr{L}_0$ to $\mathscr{C} \times \mathscr{T}\mathscr{L}_p$.

10. Time inversion

The problem of reversing the direction of time has attracted much attention in the physical literature (see, e.g. Davies, 1974). To-day's discussions of the problem are often based on the so-called time reversal operation T (see, e.g. Bjorken and Drell, 1964), originally introduced by Wigner (1932). Here, we shall define a simpler — and from a relativistic point of view more natural — operation, which we shall denote T' and call the time inversion operation.

The effect of T' on the external coordinates is to replace t by $-t$ and thus also p_4 by $-p_4$. Hence, we get the following result for the operators (24) and (30) of the basic algebra for a spinless particle:

$$\boldsymbol{p} \rightarrow \boldsymbol{p}, \ p_4 \rightarrow -p_4, \ \boldsymbol{l} \rightarrow \boldsymbol{l}, \ \boldsymbol{k} \rightarrow -\boldsymbol{k}. \tag{116}$$

The substitution (48) requires, that if T' is to be accepted as an invariance operation, then we must demand that

$$H \to -H. \tag{117}$$

This condition is certainly not fulfilled for the Hamiltonian (49). The description of a spinless particle, as developed in section 3, is thus not invariant under time inversion.

For a particle with spin we require that the internal generators be transformed similarly to (116), i.e.

$$\boldsymbol{s} \to \boldsymbol{s}, \kappa \to -\kappa. \tag{118}$$

It is now easy to verify that all three relations (116)–(118), with H as given by (86), are satisfied, if we define T' as the process, which besides transforming t into $-t$ changes the Euler angles according to the scheme:

$$\alpha \to \alpha + \pi, \beta \to \pi - \beta, \gamma \to \pi - \gamma. \tag{119}$$

This corresponds to a 2-fold rotation about the $\boldsymbol{e}_2$-axis, just as (112) corresponds to a 2-fold rotation about the $\boldsymbol{e}_3$-axis. The effect on the ζ_i operators is:

$$\zeta_1 \to -\zeta_1, \quad \zeta_2 \to \zeta_2, \quad \zeta_3 \to -\zeta_3. \tag{120}$$

The functions (17) are transformed thus:

$$\left.\begin{aligned} \theta_1 &\to \theta_3, \\ \theta_2 &\to \theta_4, \\ \theta_3 &\to -\theta_1, \\ \theta_4 &\to -\theta_2, \end{aligned}\right\} \tag{121}$$

Hence we obtain, from the explicit expressions (102):

$$\left.\begin{aligned} T'\Psi_1 \equiv \Psi_3 &= \cos\frac{\eta}{2}\left[\theta_3 - \frac{c}{|E| + m_0 c^2}(\pi_z\theta_1 + \pi_+\theta_2)\right]\exp(\iota\boldsymbol{\pi}\cdot\boldsymbol{r}/\hbar)\exp(\iota|E|t/\hbar), \\ T'\Psi_2 \equiv \Psi_4 &= \cos\frac{\eta}{2}\left[\theta_4 - \frac{c}{|E| + m_0 c^2}(\pi_-\theta_1 - \pi_z\theta_2)\right]\exp(\iota\boldsymbol{\pi}\cdot\boldsymbol{r}/\hbar)\exp(\iota|E|t/\hbar). \end{aligned}\right\} \tag{122}$$

The functions Ψ_3 and Ψ_4 may, just as well as the function $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ in (103), be used as representatives for the function space $\tilde{\Omega}$. Ψ_3 and Ψ_4 are, in fact, equal to $-\tilde{\Psi}_2$ and $\tilde{\Psi}_1$, respectively, with $-\boldsymbol{\pi}$ substituted for $\boldsymbol{\pi}$. The effect of T' on Ψ_3 and Ψ_4 is:

$$\left.\begin{aligned} T'\Psi_3 &= -\Psi_1, \\ T'\Psi_4 &= -\Psi_2. \end{aligned}\right\} \tag{123}$$

Hence, the time inversion operation converts a particle state into an anti-particle state and vice versa, the velocity of the LORENTZ frame associated with the particle being reversed during the operation.

Adding the time inversion operation to the operations of $\mathscr{T}\mathscr{L}_p$, leads to the full inhomogeneous LORENTZ group $\mathscr{T}\mathscr{L}$. The charge conjugation operation commutes with T' just as it commutes with all other elements of $\mathscr{T}\mathscr{L}$, and the full invariance group is thus found to be $\mathscr{C} \times \mathscr{T}\mathscr{L}$.

This important result justifies the introduction of T' and demonstrates the fundamental nature of this operation. To anchor it further, let us demonstrate the consistent transformation properties of the 4-vectors of our theory, with respect to space and time inversion.

A 4-vector $a_\mu = (\boldsymbol{a}, a_4)$ is a set of four quantities satisfying a relation similar to (29), viz.

$$[J_{\mu\nu}, a_\lambda] = \iota\hbar(\delta_{\mu\lambda}a_\nu - \delta_{\nu\lambda}a_\mu). \tag{124}$$

The following expressions are readily found to correspond to 4-vectors:

$$x_\mu = (\boldsymbol{r}, \text{ict}), \tag{23}$$

$$p_\mu = (\boldsymbol{p}, p_4), \tag{24}$$

$$w_\mu = (\boldsymbol{p} \times \boldsymbol{\mathsf{K}} + p_4\boldsymbol{\mathsf{s}}, -\boldsymbol{p} \cdot \boldsymbol{s}), \tag{62a}$$

$$\gamma'_\mu = \left(-\zeta'_2 \boldsymbol{s}', \frac{\iota}{i}\zeta'_3\right). \tag{125}$$

The matrices associated with the operators γ'_μ and the basis (17) are, when $\iota = i$, identical with the γ_μ matrices of the conventional theory, in the notation of DIRAC (1928) and e.g. SAKURAI (1967). The γ'_μ operators turn up in a natural manner, when (47) is multiplied from the left with ζ'_3, to give the equation

$$(m_0 c + \iota\gamma'_\mu p_\mu)\Psi = 0. \tag{126}$$

USING the properties of the P and T' operations as described above, it is easily seen that x_μ, p_μ and γ'_μ transform according to the scheme:

$$\left.\begin{aligned} P(\boldsymbol{a}, a_4) &= (-\boldsymbol{a}, a_4), \\ T'(\boldsymbol{a}, a_4) &= (\boldsymbol{a}, -a_4), \end{aligned}\right\} \tag{127}$$

whereas w_μ transforms as follows:

$$\left.\begin{aligned} P(\boldsymbol{w}, w_4) &= (\boldsymbol{w}, -w_4), \\ T'(\boldsymbol{w}, w_4) &= (-\boldsymbol{w}, w_4). \end{aligned}\right\} \tag{128}$$

These transformation properties characterize x_μ, p_μ and γ'_μ as ordinary 4-vectors, w_μ as a pseudo-4-vector. For all four vectors it holds, that

$$PT'a_\mu = T'Pa_\mu = -a_\mu. \tag{129}$$

The "strong inversion" operation $PT' = T'P$ will be the subject matter of the following section.

The possibility of defining a time inversion operation with the above properties in the case of a DIRAC particle suggests, that a similar operator may be defined for other elementary systems as well. Let us, for the moment, assume that this is possible for the electromagnetic field. This field is characterized by a 4-vector

$$A_\mu = (\boldsymbol{A}, i\varphi), \tag{130}$$

where $\boldsymbol{A}$ is the vector potential and φ the scalar potential. It is well known, that

$$P(\boldsymbol{A}, i\varphi) = (-\boldsymbol{A}, i\varphi), \tag{131}$$

and comparison with (127) makes us therefore expect, that

$$T'(\boldsymbol{A}, i\varphi) = (\boldsymbol{A}, -i\varphi). \tag{132}$$

Suppose now, that the source of A_μ is a charged DIRAC particle. The field associated with the corresponding antiparticle must then be $(\boldsymbol{A}, -i\varphi)$. In other words, a particle and its associated antiparticle must have equal, but opposite, charges.

That this is indeed the case is of course well known. The interesting thing in the present context is, that we have tied the conclusion to the properties of the full LORENTZ group, rather than to the properties of the charge conjugation operation. A more appropriate name for the latter, is, in fact, the often used alternative: the particle-antiparticle conjugation operation.

11. Strong inversion, alias the PCT-operation

Combining the operations P and T' leads to what we shall call the strong inversion operation, I. It changes x_μ into $-x_\mu$, while the EULER angles undergo the transformation corresponding to a 2-fold rotation about the $\boldsymbol{e}_1$-axis, i.e.

$$\alpha \to \alpha + \pi, \qquad \beta \to \pi - \beta, \qquad \gamma \to -\gamma. \tag{133}$$

Hence, we get:

$$\left.\begin{aligned} & p_\mu \to -p_\mu, \boldsymbol{l} \to \boldsymbol{l}, \boldsymbol{k} \to \boldsymbol{k}, \\ & \boldsymbol{s} \to \mathsf{s}, \mathsf{K} \to \mathsf{K}, \\ & H \to -H, \\ & w_\mu \to -w_\mu, \gamma'_\mu \to -\gamma'_\mu. \end{aligned}\right\} \tag{134}$$

The functions (17) are transformed thus:

$$I[\theta_1\theta_2\theta_3\theta_4)] = -\iota[\theta_3\theta_4\theta_1\theta_2], \tag{135}$$

and for the general function (108) we obtain:

$$I\sum_i \theta_i \psi_i(x_\mu) = -\iota[\theta_1\theta_2\theta_3\theta_4]\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \psi_1(-x_\mu) \\ \psi_2(-x_\mu) \\ \psi_3(-x_\mu) \\ \psi_4(-x_\mu) \end{bmatrix}. \tag{136}$$

The 4×4matrix in (136) is the matrix representative of the operator ζ'_1. It is readily identified with the matrix

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \tag{137}$$

of the conventional theory.

A comparison with e.g. Bjorken and Drell (1964) shows us now, that I has the same effect on a general wavefunction as the so-called PCT-operation. Hence, we have arrived at an alternative and simple interpretation of this fundamental operation.

The relation between the operations P, T', I and the 2-fold rotations about the three internal axes of the rotor is a nice illustration of the group theoretical fact, that the factor group of $\mathscr{T}\mathscr{L}$ with respect to the invariant subgroup $\mathscr{T}\mathscr{L}_0$ is isomorphic with the group D_2.

12. Wigner's time reversal operation

Combining the operation T' and C leads to the operation

$$T = CT', \tag{138}$$

which we shall now identify as Wigner's time reversal operation. The effect of C is to leave x_μ unaffected, while each operator of the basic algebra (and thus also H) changes its sign. The effect of T' was considered in section 10. Hence we get:

$$\left.\begin{array}{l} \boldsymbol{r}\to\boldsymbol{r},\ t\to -t, \\ \boldsymbol{p}\to -\boldsymbol{p},\ p_4\to p_4,\ H\to H, \\ \boldsymbol{l}\to -\boldsymbol{l},\ \boldsymbol{k}\to\boldsymbol{k},\ \boldsymbol{s}\to -\boldsymbol{s},\ \kappa\to\kappa, \\ \zeta_1\to\zeta_1,\ \zeta_2\to -\zeta_2,\ \zeta_3\to\zeta_3. \end{array}\right\} \quad (139)$$

The effect on the general function (108) is found by combining (121) with (109):

$$T\sum_i \theta_i \psi_i(\boldsymbol{r}, t) = [\theta_1\theta_2\theta_3\theta_4]\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} \psi_1^*(\boldsymbol{r}, -t) \\ \psi_2^*(\boldsymbol{r}, -t) \\ \psi_3^*(\boldsymbol{r}, -t) \\ \psi_4^*(\boldsymbol{r}, -t) \end{bmatrix}. \quad (140)$$

The 4×4 matrix in (140) is equal to ι times the matrix representative of s_2', and equal to the matrix $-\gamma_1\gamma_3$ of the conventional theory.

Thus, the relations (139) and (140) establish the assertion, that the complicated operation known as WIGNER's time reversal operation may be considered as a compound operation, made up of the two elementary operations C and T'.

With this result, we have seen that all the symmetry operations of the conventional theory have a simple representation within the rotor model. It is further worth-while noting, that this model leads to a clear understanding of the way in which antilinear operators enter the theory: All operations of the group $\mathscr{T}\mathscr{L}$ correspond to linear operators, the fundamental antilinear operation being the operation C.

13. Some general remarks

The rotor model as developed so far is a one-particle model, and the comparisons we have made with the conventional theory have, accordingly, not included references to discussions based on field theoretical descriptions. There is, of course, a very extensive literature on the symmetries of the quantized DIRAC field (see e.g. KEMMER et al., 1959; MUIRHEAD, 1965). This literature leaves the general impression, that an operation like the PCT operation has its roots in the connection between spin and statistics (PAULI, 1955; LÜDERS, 1957). We have no reason to doubt that this is true in general, but would like to stress that the PCT operation as it occurs in the present treatment is a very simple operation. The presence of the letter C in its designation is in fact misleading, since it appears as a genuine operation of the group $\mathcal{T}\mathcal{L}$, of which C is not a member.

As far as C itself is concerned, we may obtain a clearer understanding of its nature by tying it to the presence of the indicator ι, which we have carried through as an unassigned quantity. This is, however, best discussed elsewhere.

Finally, we shall consider the ambiguity in sign of the first term of the Hamiltonian (86). We have so far developed the theory with the plus sign, but it may equally well be developed with the minus sign. The only difference in the resulting wavefunctions is, that the $(\boldsymbol{r},t)$ dependent parts in (102) and (103) are interchanged. In the conventional theory one could talk about an interchange of the large and the small components of the wavefunction, and there would be no basis for believing that one had obtained anything but an alternative description of the same physical situation.

If, however, one adopts the rotor model, then there is no way of transforming the time-dependent wavefunctions corresponding to the two different signs into each other, and the two Hamiltonians must be considered as physically different, i.e. they must be associated with two different types of Dirac particles. It is, however, easy to see that the two types of particles will behave similarly in an electromagnetic field; the type of interaction which can distinguish between them must be of a different nature.

We are, of course, unable to settle the question as to whether such an interaction exists or not. If it does not, then one is free to choose either sign in the Hamiltonian. If, however, is does exist, then one might perhaps imagine a connection to the electron-muon problem.

14. Conclusion

The discussion of sections 7–12 illustrates the type of natural interpretation one obtains by considering a Dirac particle as a quantum mechanical rotor. The preceding sections taught us, that the only type of behaviour that a relativistic, quantum mechanical rotor can adopt, is that of a Dirac particle.

Thus, we arrive at the conclusion, that the Dirac particle and the quantum mechanical rotor are identical dynamical systems. In other words: a Dirac particle is neither more nor less than a particle, for which it is possible to talk about an orientation in space.

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